ON THE REPRESENTATIONS OF DIFFERENTIALS IN FUNCTIONAL RINGS AND THEIR APPLICATIONS

ANATOLIY K. PRYKARPATSKY AND DENIS BLACKMORE

Take the ring $\mathcal{K} := \mathbb{R}\{\{x,t\}\}$, $(x,t) \in \mathbb{R}^2$, of convergent germs of real-valued smooth functions from $C^{(\infty)}(\mathbb{R}^2;\mathbb{R})$ and construct the associated [1] differential polynomial ring $\mathcal{K}\{u\} := \mathcal{K}[\Theta u]$ with respect to a functional variable u, where Θ denotes the standard monoid of all operators generated by commuting differentiations $\partial/\partial x := D_x$ and $\partial/\partial t$. The ideal $I\{u\} \subset \mathcal{K}\{u\}$ is called differential if the condition $I\{u\} = \Theta I\{u\}$ holds.

Consider now the additional differentiation

$$D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}, \tag{1}$$

depending on the functional variable u, which satisfies the Lie-algebraic commutator condition

$$[D_x, D_t] = (D_x u) D_x, \tag{2}$$

for all $(x,t) \in \mathbb{R}^2$. As a simple consequence of (2) the following general (suitably normalized) *representation* of the differentiation (1)

$$D_t = \partial/\partial t + u\partial/\partial x \tag{3}$$

in the differential ring $\mathcal{K}\{u\}$ holds. Impose now on the differentiation (1) a new algebraic constraint

$$D_t^{N-1} u = \bar{z}, \qquad D_t \bar{z} = 0, \tag{4}$$

defining for all natural $N \in \mathbb{N}$ some smooth functional set (or "*manifold*") $\mathcal{M}^{(N)}$ of functions $u \in \mathbb{R}\{\{x,t\}\}$, and which allows to reduce naturally the initial ring $\mathcal{K}\{u\}$ to the basic ring $\mathcal{K}\{u\}|_{\mathcal{M}_{(N)}} \subseteq \mathbb{R}\{\{x,t\}\}$. In this case the following natural problem of constructing the corresponding representation of differentiation (1) arises: *to find an equivalent linear representation of the reduced differentiation* $D_t|_{\mathcal{M}_{(N)}} : \mathbb{R}^{p(N)}\{\{x,t\}\} \to \mathbb{R}^{p(N)}\{\{x,t\}\}$ *in the functional vector space* $\mathbb{R}^{p(N)}\{\{x,t\}\}$ *for some specially chosen integer dimension* $p(N) \in \mathbb{Z}_+$.

We have shown that for arbitrary $N \geq 2$ this problem is completely analytically solvable by means of the differential-algebraic tools, devised in [2, 3], giving rise to the corresponding Lax type integrability of the generalized Riemann type hydrodynamical system (4). Moreover, the same problem is also solvable for the more complicated constraints

$$D_t^{N-1} u = \bar{z}, \qquad D_t \bar{z}_x^2 = 0, \tag{5}$$

equivalent to a generalized Riemann type hydrodynamic flows, and

$$D_t u - D_x^3 u = 0, \quad D_x D_t u - u = 0, \tag{6}$$

equivalent to the Lax type integrable nonlinear Korteweg-de Vries and Ostrovsky-Vakhnenko dynamical systems.

References

[1] Ritt J.F. Differential algebra. AMS-Colloqium Publications, vol. XXXIII, New York, NY, Dover Publ., 1966

[2] Prykarpatsky A.K., Artemovych O.D., Popowicz Z. and Pavlov M.V. Differential-algebraic integrability analysis of the generalized Riemann type and Korteweg–de Vries hydrodynamical equations. J. Phys. A: Math. Theor. 43 (2010) 295205 (13pp)

[3] Blackmore D., Prykarpatsky A.K. and Samoylenko V.Hr. Nonlinear dynamical systems of mathematical physics: spectral and differential-geometrical integrability analysis. World Scientific Publ., NJ, USA, 2011

AGH University of Svience and Technology of Krakow, Poland, and the Ivan Franko State Pedagogical University, Drohobych, Lviv region, Ukraine

The New Jersey Institute of Technology, Newark, 07102, NJ USA

Date: present.