ON LOCAL CONVEXITY OF NONLINEAR MAPPINGS IN BANACH SPACES

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Abstract. The sufficient conditions for a smooth nonlinear Lipschitzian mapping of Banach spaces to be locally convex are formulated.

1. Introduction

Local convexity of nonlinear mappings of Banach spaces is important for many branches of applied mathematics [13, 14], in particular, in the theory of nonlinear differential-operator equations, optimization and control theory etc. Some of interesting properties of such locally convex mappings were studied by many theoretical [1, 2, 3, 4, 8] and applied [6, 7, 14] mathematicians.

Recall that a nonlinear continuous mappping $f : X \to Y$ of Banach spaces X and Y *is called to be locally convexive, if for any point* $a \in X$ there exists a ball $B_\varepsilon(a) \subset X$ of radius $\varepsilon > 0$, such that its image $f(B_\varepsilon(a)) \subset Y$ is convex. Concerning the special case of a differentiable mapping $f : X \to Y$ of Hilbert spaces, the property of local convexity holds if the Frechèt derivative $f'(x) : X \to Y$ is Lipschitzian in a closed ball $B_r(a) \subset X$ of radius $r > 0$ centered at point $a \in X$ and the linear mapping $f'(a) : X \to Y$ is surjective. A proof of this statement is based on the strong convexity of the ball $B_r(a) \subset X$ in the Hilbert space X. The local convexity problem for a nonlinear differentiable mapping $f : X \to Y$ of Banach spaces needs more subtle techniques and its analysis is done only for the case of reflexive Banach spaces.

2. The Hilbert space case

Since this is not a case for the nonlinear mappings mappings of Banach spaces, a problem arises: to construct at least sufficient conditions for a nonlinear smooth mapping $f : X \to Y$ of a Banach space X into a Banach space Y to be locally convex.

For convenience of entering into the problem, we will make a sketch of the local convexity proof for a nonlinear smooth mapping $f : X \to Y$ of Hilbert spaces.

Proposition 2.1. *Let $f : X \to Y$ be a nonlinear differentiable mapping of Hilbert spaces whose Frechèt derivative $f'(x) : X \to Y$, $x \in B_r(a)$, in a ball $B_r(a) \subset X$ centered at point $a \in X$, is Lipschitzian with a constant $L > 0$, the linear mapping $f'(a) : X \to Y$ is surjective and the adjoint mapping $f'(a)^* : Y \to X$ satisfies the condition $||f'(a)^*|| \geq \nu$ for some positive $\nu > 0$. Then for any $\varepsilon < \min\{r, \nu/(2L)$ the image $F_\varepsilon(a) := f(B_\varepsilon(a)) \subset Y$ is convex.*

To prove Proposition 2.1 above, it is useful to state the following elementary enough lemmas, based both on the Taylor expansion of the differentiable mapping $f : X \to Y$ at point $x_0 \in B_\varepsilon(a) \subset B_r(a)$ and on the triangle and parallelogram properties of the norm $|| \cdot ||$ in Hilbert spaces.

Lemma 2.2. *Let a mapping $f'(x) : X \to Y$ be L-Lipschitzian in a ball $B_\rho(x_0) \subset B_\varepsilon(a)$ of radius $\rho > 0$, centered at point $x_0 := (x_1 + x_2)/2 \in B_\varepsilon(a)$ for arbitrarily chosen points $x_1, x_2 \in B_\varepsilon(a)$. Then there exists such a positive constant $\mu > 0$ that the norm $||f'(x)^* y|| \geq \mu ||y||$ in the ball $B_\rho(x_0) \subset X$ for all $y \in Y$, there holds the estimation $||f(x_0) - y_0|| \leq \rho\mu$ for $y_0 := (y_1 + y_2)/2$, $y_1 := f(x_1), y_2 := f(x_2)$ and the equation $f(x) = y_0$ possesses a solution $\bar{x} \in B_\rho(x_0)$ such that $||\bar{x} - x_0|| \leq \mu^{-1} ||f(x_0) - y_0||$.*

Date: present.

Key words and phrases. Nonlinear mapping, local convexity property, reflexive Banach spaces.

Proof. Really, the following Taylor expansions at point $x_0 \in B_\varepsilon(a)$ hold:

$$
\begin{aligned}
y_1 &= f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \epsilon_1, \\
y_2 &= f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \epsilon_2,
\end{aligned} \tag{2.1}
$$

where $||\epsilon_j|| \le \frac{L}{2}||x_j - x_0||^2 = \frac{L}{8}||x_1 - x_2||^2, j = \overline{1,2}$, as the mapping $f'(x) : X \to Y$ is L-Lipschitzian. From (2.1) one obtains easily that

$$y_0 = f(x_0) + \epsilon_0, \tag{2.2}$$

where, evidently, $||\epsilon_0|| \le (||\epsilon_1|| + ||\epsilon_2||)/2 \le \frac{L}{8}||x_1 - x_2||^2$. Moreover, owing to the Lipschitzian property of the Frechèt derivative $f'(x) : X \to Y$, one can obtain the following inequality:

$$
\begin{aligned}
||f'(x)^* y|| &= ||f'(x)^* y - f'(a)^* y + f'(a)^* y|| \ge \\
&\ge ||f'(a)^* y|| - ||f'(x)^* y - f'(a)^* y|| \ge \\
&\ge \nu||y|| - L||x - a||||y|| \ge (\nu - L\varepsilon)||y|| := \mu||y||,
\end{aligned} \tag{2.3}
$$

for $\mu = (\nu - L\varepsilon) > 0$, as the norm $||x - a|| \le \varepsilon$. This, in particular, means that the adjoint mapping $f'(x)^* : Y \to X$ is invertible, defined on the whole Hilbert space Y and the norm of its inverse mapping $(f'(x)^*)^{-1} : X \to Y$ is bounded in the ball $B_\varepsilon(a)$ by the value $1/\mu$.

The following inequality, based on (2.2) and the additionally assumed conditions $\nu > 2L\varepsilon$ and $\rho := \frac{L}{8\varepsilon}||x_1 - x_2||^2$, holds:

$$
\begin{aligned}
||f(x_0) - y_0|| &= ||\epsilon_0|| \le \frac{L}{8}||x_1 - x_2||^2 = \\
&= L\rho\varepsilon \le \rho(\nu - L\varepsilon) - \rho(\nu - 2L\varepsilon) \le \\
&\le \rho(\nu - L\varepsilon) = \rho\mu.
\end{aligned} \tag{2.4}
$$

Denote now by $\bar{x} \in B_\varepsilon(a)$ a point satisfying the equation $y_0 = f(\bar{x})$, whose existence is owing to the standard implicit function theorem [10, 11], and by $\bar{y} := (f'(\tilde{x})^*)^{-1}(\bar{x} - x_0) \in Y$, where a point $\tilde{x} := x_0 + \delta(\bar{x} - x_0) \in B_\rho(x_0)$ for some $\delta \in (0,1)$, is defined by means of the Taylor expansion as

$$f(\bar{x}) - f(x_0) = f'(\tilde{x})(\bar{x} - x_0), \tag{2.5}$$

where, evidently, $||\tilde{x} - x_0|| \le \delta||\bar{x} - x_0|| \le ||\bar{x} - x_0||$. Then one can obtain that

$$
\begin{aligned}
||\bar{x} - x_0||^2 &= |(x - x_0, f'(\tilde{x})^* \bar{y})| = |(f'(\tilde{x})(\bar{x} - x_0), \bar{y})| = \\
&= |(f(\bar{x}) - f(x_0), \bar{y})| = |(y_0 - f(x_0), (f'(\tilde{x})^*)^{-1}(\bar{x} - a)| \le \\
&\le ||y_0 - f(x_0)||\ ||(f'(\tilde{x})^*)^{-1}||\ ||\bar{x} - x_0|| \le \\
&\le ||y_0 - f(x_0)||\ ||\bar{x} - x_0||/\mu,
\end{aligned}
$$

yielding the searched for inequality

$$||\bar{x} - x_0|| \le ||y_0 - f(x_0)||/\mu \tag{2.6}$$

and proving the Lemma.

Lemma 2.3. *For arbitrarily chosen points $x_1, x_2 \in B_\varepsilon(a)$ the whole ball $B_\rho(x_0)$ of radius $\rho = ||x_1 - x_2||^2/(8\varepsilon) \le \varepsilon$, centered at point $x_0 := (x_1 + x_2)/2 \in B_\varepsilon(a)$, belongs to the ball $B_\varepsilon(a)$.*

□

Proof. Consider for this the following triangle inequality and the related parallelogram identity for any point $x \in B_\rho(x_0)$:

$$
\begin{aligned}
||x - a|| &= ||(x - x_0) + (x_0 - a)|| \le ||x - x_0|| + ||x_0 - a|| = \\
&= ||x - x_0|| + ||(x_1 - a)/2 + (x_2 - a)/2|| = \\
&= ||x - x_0|| + [2(||x_1 - a||^2/4 + ||x_2 - a||^2/4) - ||x_1 - x_2||^2/4]^{1/2} \le \\
&\le \rho + (\varepsilon^2 - ||x_1 - x_2||^2/4)^{1/2}.
\end{aligned} \tag{2.7}
$$

For the righthand side of (2.7) to be equal or less of $\varepsilon > 0$, it is enough to take such positive $\rho \le \varepsilon$ that

$$\rho + (\varepsilon^2 - ||x_1 - x_2||^2/4)^{1/2} \le \varepsilon. \tag{2.8}$$

This means that the following inequality should be satisfied:

$$\rho^2 \geq 2\varepsilon\rho - ||x_1 - x_2||^2/4. \tag{2.9}$$

The preceding choice $\rho = ||x_1 - x_2||^2/(8\varepsilon)$ satisfies (2.9) in the evident form $\rho^2 \geq 0$, thereby proving the Lemma. □

Proof. Now, based on Lemmas 2.2 and 2.3, it is easy to observe from (2.4) and (2.6) that a point $\bar{x} \in B_r(a)$, satisfying the equation $y_0 = f(\bar{x})$, belongs to the ball $B_\rho(x_0)$:

$$||\bar{x} - x_0|| \leq ||y_0 - f(x_0)||/\mu \leq \rho\mu/\mu = \rho, \tag{2.10}$$

giving rise to the imbedding $\bar{x} \in B_\varepsilon(a)$. The latter proves the local convexity property of our Proposition 2.1, that is the image $F_\varepsilon(a) := f(B_\varepsilon(a)) \subset Y$ is convex. □

3. The Banach space case

Now we will proceed to studying the case of a nonlinear Frechèt differentiable Lipschitzian mapping $f : X \to Y$ of Banach spaces. Before doing this let us formulate some important lemmas holding in Banach spaces.

Lemma 3.1. *(à la S. Mazur) Every convex, closed and bounded set in a reflexive Banach space is weakly compact.*

This theorem is charactersitic for reflexive Banach spaces and means, in particular, that a closed ball $B_\varepsilon(a) \subset X$ is weakly compact.

Lemma 3.2. *(Michael) Let X and Y be Banach spaces and $A : X \to Y$ be a surjective and closed linear operator. Then there exists an invertible from the right multi-valued operator $\tilde{A}^{-1} : Y \to X/\ker A$ and its continuous anti-symmetric selection $S_{(A)} : Y \to X$, such that the following equality $A \cdot S_{(A)}(y) = y$ holds for all $y \in Y$.*

Proof. A proof is standard [5] and based on the classical Banach closed graph theorem [9] and a direct successive iterative construction of an inverse selection $S_{(A)} : Y \to X$. □

Lemma 3.3. *Let a nonlinear differentiable mapping $f : X \to Y$, where X is a reflexive Banach space, be such that the Frechèt derivative $f'(x) : X \to Y$ on a sphere $B_\varepsilon(a) \subset X$ of radius $\varepsilon > 0$ is Lipschitzian with a constant $L > 0$, the linear mapping $f'(a) : X \to Y$ is closed and surjective. Suppose also that the inequality $\varepsilon < k(S_{(f'(a))})/(2L)$ holds, where*

$$k(S_{(f'(a))}) := \sup_{y\in Y} \|y\|_Y^{-1} \inf_{x\in X} \{\|x\|_X : f'(a)x = y\}. \tag{3.1}$$

Then a nonlinear mapping

$$B_\varepsilon(a) \ni x \to S_{(f'(a))}[f(x) - f(a)] + a := \alpha_f(x) \in X \tag{3.2}$$

is a injective on the ball $B_\varepsilon(a)$, satisfying the estimation

$$||f(\alpha_f(x)) - f(x)|| \leq L||\alpha_f(x) - a||^2/2 \tag{3.3}$$

for all $x \in B_\varepsilon(a)$.

Proof. A proof easily holds from the Taylor expansion applied to the mapping $f : X \to Y$ at point $a \in X$ and from condition (3.2). □

Then, based on Lemmas 3.1 - 3.3, the following final proposition holds.

Proposition 3.4. *For any $0 < \varepsilon < \min\{r, k(S_{(f'(a))})/(2L)\}$ the image*

$$F_\varepsilon(a) = f(B_\varepsilon(a)) := \{f(x) \in Y : x \in B_\varepsilon(a)\} \tag{3.4}$$

of the ball $B_\varepsilon(a) \subset X$ is a convex set.

Proof. As the ball $B_\varepsilon(a) \subset X$ is weakly compact, the set $\tilde{F}_\varepsilon(a) := conv F_\varepsilon(a) \subset Y$ is weakly compact too. For the proposition to be proved it is enough to show that $\tilde{F}_\varepsilon(a) = F_\varepsilon(a)$. □

The local convexity condition 3.4 formulated above, to the regret, appears to be from application point of view not enough effective, and needs additional more detailed analysis of the problem.

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