Nonclasical solutions to systems of conservation laws with the delta-shape singularities

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March 23, 2012

It is well known that there are "nonclassical" situations where the Cauchy problem for a system of conservation laws does not possess a classical L^∞-*generalized solution* (weak solution) solution or possesses it for some particular initial data. In order to solve the Cauchy problem in this "nonclassical" situation, it is necessary to introduce new singular solutions with the *delta shape singularities*. Problems related to such solutions have been intensively studied in the last twenty years. These solutions are connected with transport and concentration processes of some quantities, for example, mass, momentum, energy, etc.

The above-mentioned singular solutions do not satisfy the standard (L^∞) integral identities. To deal with them we must:

(i) to develop a *special technique*,

(ii) *discover a proper notion of weak solution*, i.e., to define in which sense a distributional solution satisfies a nonlinear system.

We study some important systems of conservation laws admitting such solutions. Among these systems are

(a) The zero-pressure gas dynamics with the energy conservation law

$$\begin{aligned} \rho_t + \nabla\cdot(\rho U) &= 0, \\ (\rho U)_t + \nabla\cdot(\rho U\otimes U) &= 0, \\ \Big(\frac{\rho|U|^2}{2}+H\Big)_t + \nabla\cdot\Big(\Big(\frac{\rho|U|^2}{2}+H\Big)U\Big) &= 0, \end{aligned}$$

where ρ is the density, U is the velocity, $H(x,t)$ is the internal energy, $\otimes$ is the tensor product of vectors.

(b) A new type of systems of conservation laws (admitting δ-shocks)

$$(u_j)_t + \big(u_j\, f_j(\mu_1 u_1 + \cdots + \mu_n u_n)\big)_x = 0, \quad x\in\mathbb{R}, \quad t\ge 0,$$

where $f_j(\cdot)$ is a smooth function, μ_j is a constant, $j=1,2,\dots,n$. This class includes some *Temple type system*, in particular, the system of *nonlinear chromatography* and the system for *isotachophoresis*.

(c) The Navier-Stockes granular hydrodynamics

$$\begin{aligned}\rho_t + \nabla \cdot (\rho U) &= 0,\\ (\rho U)_t + \nabla \cdot (\rho U \otimes U + I\,\rho T) &= 0,\\ T_t + \nabla \cdot (UT) &= (2-\gamma)T\,\nabla \cdot U - \Lambda \rho T^{3/2},\end{aligned}$$

where ρ is the gas density, U is the velocity, T is the temperature, $p = \rho T$ is the pressure; γ is the adiabatic index (if $n = 2$ then $\gamma = 2$, and if $n = 3$, then $\gamma = 5/3$), Λ is a constant connected with the energy of collision processes (which can be calculated in the framework of the kinetic theory).

We introduce integral identities to define delta-shock solutions of the above systems and derive the corresponding Rankine-Hugoniot conditions. In contrast to the case of shock waves, the wave fronts related with these solutions *carry mass, momentum and energy.* We derive balance laws describing mass, momentum, and energy transport from the area outside the delta-shock wave front onto this front. The Riemann problems admitting delta-shock are solved.