On the relative equilibrium configuration in the planar six-body problem

Agnieszka Siluszyk, Dorota Kozak-Superson

Siedlce University of Natural Science and Humanitities,
Institute of Mathematics and Physics, Poland

sil_a@uph.edu.pl, kdorota@uph.edu.pl

INTRODUCTION

Central configurations in the N-body problem are such configurations that the total Newtonian acceleration on each body is equal to a constant multiplied by the position vector of this body with respect to the center of mass of the configuration (see e.g., [1]).

We are interested in central configurations because they help to explain homographic solutions of the N-body problem. Central configurations also appear as a key point when we study the topology of the set of points of the phase space having energy *h* and angular momentum *c*. Moulton in 1910 characterized the number of collinear central configurations by showing that there exist exactly *N!/2* classes of central configurations of the N-body problem for a given set of positive masses (see e.g., [1]). The number of classes of planar central configurations of the N-body problem for an arbitrary given set of positive masses has been only solved for *N = 3*. In this case L.Euler has found a collinear relative equilibrium, and J.L.Lagrange has found central configurations as two equilateral triangles (see [1] and the bibliography therein).

In the 90-ies B.Elmabsout [2] and E.A.Grebenicov [3] have proved that besides of the class of gravitational models in the inertial barycentric system (the so-called gravitational model of Lagrange-Wintner), there exists a new class of gravitational models, i.e., the class of gravitational models in non inertial frames (the so-called gravitational model of Grebenicov-Elmabsout, we denote it by GE(m_0;N)). They have proved that there exists a relative equilibrium configuration in the (N + 1)-body problem with N bodies with equal masses located at the vertices of a regular N-gon, and with a body of non-zero mass situated in the center of the polygon. In 1991 B.Elmabsout has stated necessary and sufficient conditions for the existence of a relative equilibrium configuration of the (N+1)-body problem for the Grebenicov-Elmabsout models when *N* material particles are located at the vertices of *p* regular *n*-gons centered at a given point-mass m_0, with the bodies on the same *n*-gon having equal masses, and therefore $N = p\cdot n$ [4]. (We will denote this class of models by GE(m_0; p, n)). In our paper we study the six-body problem, where six bodies interact according to the Newtonian attraction. We deal with a central configuration of six point-masses (m_i; q_i) ($q_i \in R^2$, $m_i \in R^+$, i =1, …, 6) located at the edges of a segment of length 2*q* with the rest four point-masses at the vertices of a square with the side $q'\sqrt{2}$. The segment is contained into a diagonal of the square and both have the same center of symmetry with *m =0*.

During the last two decades in the N-body problem a series of papers on central configurations of type GE(m_0; p, n) with *p* = 1; 2; 3and *n* = 2; 3; 4; 5 arose (E.A.Grebenicov, N.I.Zemtsova, A.Siluszyk, E.V.Ihsanov, D.Kozak-Skoworodkin). The problem of existence, finiteness and the evaluation of the number of central configurations in an asymmetric N-body problem for *N* = 5; 7 has been stated by A.Siluszyk [5, 6].

DEFINITION AND MOTION'S EQUATIONS OF N BODIES

We begin by defining central configurations of the Newtonian N-body problem in the plane. The N-body problem in celestial mechanics is concerned with the dynamics of *N* material points (m_1, q_1), (m_2, q_2), ...,(m_N, q_N), $m_i \in R^+$, $i = 1, \ldots, N$ (the classical Newtonian problem), moving according to Newton's laws of motion [1]:

$$m_i \cdot \ddot{q}_i = \frac{\partial U}{\partial q_i}, \tag{2.1}$$

where *U* is the potential

$$U = \sum_{1\le i<j\le N} \frac{m_i \cdot m_j}{|q_i - q_j|}. \tag{2.2}$$

Here the gravitational constant is taken equal to 1. The center of mass $\frac{1}{m}\sum_{i=1}^{N} m_i q_i$ with $m = m_1 + m_2 + \ldots + m_N$ the total mass is supposed to coincide with the origin of the inertial system. In this case the system is called the *inertial barycentric system*. By a central configuration in a barycentric frame of reference of the *N* material points (m_i, q_i) we understand a configuration $q \in R^{2N}$ that satisfies the equation [2]

$$\nabla U = \sigma \nabla I, \quad 1 \le i \le N, \tag{2.3}$$

where σ is a Lagrange multiplier and $I = \sum_{i=1}^{N} m_i |q_i|^2$ neans the moment of inertia. Due to homogeneity of functions *U* and *I* one has that $\sigma = -\frac{U}{2I}$.

In other words (2.3) can be rewritten as

$$I\frac{\partial U}{\partial q_i} = -\frac{1}{2} U \frac{\partial I}{\partial q_i}, \tag{2.4}$$

which implies that a central configuration represents a critical points of the function IU^2. Due to homogeneity of this function one has that if (m_i,q_i) is a central configuration then simultaneously (m_i,αq_i) and (m_i,Rq_i) are also central configuration for any real α≠0 and *R*(2). We shall not distinguish them. A solution *q=q(t)* of the *N* material points is called *homographic* in the barycentric frame of reference if the configuration of the bodies remains similar to itself at all times *t*. By this we mean that there exist a scalar *r=r(t)>0* and an orthogonal matrix $\Omega_2(t)$ such that for every $i = 1, \ldots, N$ and *t*

$$q_i(t) = r(t) \cdot \Omega_2(t) \cdot q_i^0, \tag{2.5}$$

where q_i^0 denotes q_i at some initial instant $t=t_0$. There are two limiting types of homographic solutions; first, when the configuration is dilating without rotation (i.e. when $\Omega_2(t)$ is the identity matrix). Such solutions we call homothetic. The other appears, when the configuration is rotating without dilatation, (i.e. *R(t)=1*). This particular homographic solution is given by

$$q_i = \Omega_2(t) \cdot q_i^0 \tag{2.6}$$

and is called a *relative equilibrium*. In this case the system rotates around the center of mass with constant angular velocity and constant mutual distances, when *t* varies.

A GEOMETRICAL INTERPRETATION OF SIX-BODY MODEL

Now in the barycentric frame of reference we consider the planar motion of six interacting material points.

We are concerned with the existence problems in two cases: namely (0; 2; 4) and (0; 4; 2). In the first case for *0 < q < q'*, we assume, that the "interior" two-points have equal masses, say *m*, whilst the "exterior" four bodies have the masses m_3, m_4, m_5 and m_6 enumerated clockwise. In the second case, for $0 < q_0 < q$, we suggest that four bodies having the masses m_3, m_4, m_5 and m_6 are "interior", whereas the rest two-points with mass *m* are "exterior" .

Let: $r_{ij}: R^2 - \Delta \to R,\ \Delta = \bigcup \Delta_{ij},\ \Delta_{ij} = \{q \in M:\ q_i = q_j\}$ and $r_{ij} = |q_i - q_j|$ be the distance between the i^{th} and j^{th} bodies. In our case for all *j,k*=1,2 and *q*= | *Om* | , *q'*= | Om_{2+j} | = | Om_{4+i} | we have the following formulas expressing distances between the material points

$$\begin{cases} r_{j,k} &= 2q\left|\sin\frac{\pi(k-j)}{2}\right|, \\ r_{k,2+j} &= \sqrt{q^2+q'^2-2q\,q'\sin\frac{2\pi(j-k)}{2}}, \\ r_{k,4+j} &= \sqrt{q^2+q'^2+2q\,q'\cos\frac{2\pi(j-k)}{2}}, \\ r_{2+j,2+k} &= 2q'\left|\sin\frac{\pi(k-j)}{2}\right|, \\ r_{2+j,4+k} &= 2q'\left|\sin(\frac{2\pi(k-j)}{4}+\frac{\pi}{4})\right|, \end{cases} \tag{3.1}$$

Moreover $r_{2+j,2+k} = r_{4+j,4+k}$. Due the symmetry of the (0,2,4) and (0,4,2) we can get the invariance with respect to rotations in R^2. The following results were established by D.Bang and B.Elmabsout

Theorem 3.1. [4] *Let Π(n, m, q) and Π(n, m.',q') be two concentric regular polygons with radii |q| and |q'| respectively. Assume that all masses on the first polygon are equal to m, whereas the second polygon consists of the bodies with masses m'. Then Π(n, m, q) and Π(n, m', q') form a relative equilibrium configuration if and only if they form a homothetic configuration or if the internal polygon is similar to the external polygon with an angle equal to π/n.*

Theorem 3.2. [4] *For given masses (m_1,...,m_p) and a mass m (the masses are fixed), there exists a central equilibrium configuration where the masses are at the vertices of p..-homothetic n-gons centered around the mass m. Moreover, when n verifies n>a(n,α), it is possible to choose the order of the masses. In the case of a Newtonian potential (α=1/2), the last inequality holds if and only if* n≤*472.*

Theorem 3.3. *Given real numbers m>0 and 0<q'<q a central configuration of the type (0; 4; 2) exists if and only if the masses m_3, m_4, m_5 and m_6 satisfy the following two conditions:*

(1) $m_3=m_5$ and $m_4=m_6$
(3.2)
(2) $\kappa_1 m+\kappa_2 m_3+\kappa_3 m_4=0$
where the coefficients κ_1, κ_2, κ_3 depend only on radii q, q' and are defined by the formulas:

$$\begin{cases} \kappa_1 &= \frac{1}{4q^3} - \frac{2}{(q^2+q'^2)^{\frac{3}{2}}}, \\ \kappa_2 &= \frac{2\sqrt{2}-1}{4q'^3} - \frac{(-\frac{1}{4q'^3}+\frac{2}{(q^2+q'^2)^{\frac{3}{2}}})(\frac{\sqrt{(q-q')^2}}{q'(q-q')^3} - \frac{(q-q')(4q^2+3qq'+q'^2)}{4q^3q'(q+q')^2})}{\frac{1}{4q^3}-\frac{2}{(q^2+q'^2)^{\frac{3}{2}}}}, \\ \kappa_3 &= \frac{1-2\sqrt{2}}{4q'^3} + \frac{(\frac{\sqrt{(q-q')^2}}{(q-q')^3} - \frac{(q-q')(4q^2+3qq'+q'^2)}{4q^3(q+q')^2})}{q'(\frac{1}{4q^3}-\frac{2}{(q^2+q'^2)^{\frac{3}{2}}})}. \\ & (\frac{1}{((q-q')^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{2}q'^3} + \frac{1}{(q+q')^3} + \frac{q'}{q}(-\frac{1}{((q-q')^2)^{\frac{3}{2}}} + \frac{1}{(q+q')^3})) \end{cases} \tag{3.3}$$

We start by noting that for the N-body problem, equation (2.4) of the central configurations is a set of 3*N* algebraic equations with the general form:

$$\sum_{j=1,i\neq j}^{N} \frac{q_i - q_j}{r_{ij}^3} m_j - \frac{U}{I}\cdot q_i = 0, \quad i = 1,\ldots,N \tag{3.4}$$

For this system, expression -U/I means the coefficient of similarity and it is a complicated function of the coordinates of the system *N* bodies. For the first time this coefficient has been described by A.Wintner [1], so the each equation of the system (3.4) has nonlinear form, moreover it contains (3*N*) variables (x_i, y_i, z_i) and *N* of parameters. System of equations (3.4) expresses the necessary and sufficient conditions of the existence of central configurations in the N-body problem.

Proof. (=>) Consider the system (3.4) and put $q_i = (x_i, y_i, 0)$ for *i=1,...6*. If we multiply the first equation of the system (3.4) by y_i, the second - by x_i and take their sum, then as a result we obtain a new system of algebraic equations (see [8]):

$$\sum_{j=1,i\neq j}^{6} m_j \frac{\langle q_j, q_i\rangle}{r_{ij}^3} = 0, \quad i,j = 1,\ldots,6 \tag{3.5}$$

It is worth noting that the system (3.5) contains six equations and now it can be considered as a homogenous linear system with respect to the variables *m*, m_3, ...,m_6. In this way we obtain the following equation

$$A^{(6)} \cdot (m_1, m_2, \ldots, m_6)^T = 0. \tag{3.6}$$

where the skew-symmetric matrix $A^{(6)}$ has elements $a_{ij} = \frac{x_j y_i - x_i y_j}{r_{ij}^3}$ for *i=1,...,6*. We know that a skew-symmetric matrix always has an order (2*N*). Moreover when *N* is odd the determinant of the skew symmetric matrix always is equal zero.

In a mathematical sense it means that for *N*-body problem exists a nonzero solution of the system (3.5) under consideration of the masses m_i. In this case the number of bodies is even, but taking into account symmetry in model (0,4,2) we get that for all *q*, *q'* $\det A^{(6)}=0$. Above condition is necessary for the existence central configuration of N-body problem, and in particular configuration of type (0,4,2). So a solution exists, independently on parameters *q* and *q'*. Write the vector equation (3.6) in more detailed form (a left four equations are identical, so we omit them), then:

$$\begin{cases} -\alpha_1 m_3 + \alpha_1 m_4 &= 0, \\ -\alpha_2 m_5 + \alpha_2 m_6 &= 0. \end{cases} \tag{3.7}$$

Where $\alpha_1 = \frac{qq'}{\sqrt{q^2+q'^2}^3}$ and $\alpha_2 = \frac{1}{q'\sqrt{2}}$.

Since, the vector equation (3.6) is fulfilled if masses m_3, m_5 and masses m_4, m_6 are equal, we denote them (for fair the next calculations) by μ_2 and μ_3, respectively. Let *ω* be the angular velocity of the rotation around the *Oz* axis of the orthonormal frame (*Oxyz*). The motion of *N* bodies in the plane barycentric frame of references (*Oxy*) is governed by the differential equations:

$$\begin{cases} \frac{d^2x_i}{dt^2} - 2\omega\frac{dy_i}{dt} &= \omega^2 x_i - \sum_{j=1,j\neq i}^{N} m_j \frac{x_i - x_j}{r_{ij}^3}, \\ \frac{d^2y_i}{dt^2} + 2\omega\frac{dx_i}{dt} &= \omega^2 y_i - \sum_{j=1,j\neq i}^{N} m_j \frac{y_i - y_j}{r_{ij}^3}. \end{cases} \tag{3.8}$$

Let (x_i, y_i, 0) be coordinate of the material point m_i in rotating frame of references *Oxy*. Then the following relations are true:

$$\begin{cases} \omega^2 x_i &= \sum_{j=1,j\neq i}^{6} m_j \frac{x_i - x_j}{r_{ij}^3}, \\ \omega^2 y_i &= \sum_{j=1,j\neq i}^{6} m_j \frac{y_i - y_j}{r_{ij}^3}, \end{cases} \tag{3.9}$$

for all *i=1,...,6*. We rewrite this system as

$$(x_i + \mathbf{i}y_i)\omega^2 = \sum_{j=1,j\neq i}^{6} m_j \frac{(x_i + \mathbf{i}y_i) - (x_j + \mathbf{i}y_j)}{r_{ij}^3}.$$

If we note $q_i = x_i + iy_i$ then we obtain

$$\omega^2 = \frac{1}{q_i} \sum_{j=1,j\neq i}^{6} m_j \frac{q_i - q_j}{r_{ij}^3}. \tag{3.10}$$

For the existence of the central configuration the following equalities are necessary holds

$$\omega_1 = \omega_2 = \ldots = \omega_i. \tag{3.11}$$

where ω_i denotes the angular velocity of the material point m_i in the rotation of the orthonormal frame *Oxy*. Consider equation (3.10) in more detailed form, that is for *k=1,2* we will obtain

$$\omega^2 = \frac{1}{q_k}(\mu_1 \sum_{j=1,j\neq k}^{2} \frac{q_k - q_j}{|q_k - q_j|^3} + \mu_2 \sum_{j=3,j\neq k}^{4} \frac{q_k - q'_j}{|q_k - q'_j|^3} + \mu_3 \sum_{j=5,j\neq k}^{6} \frac{q_k - q'_j}{|q_k - q'_j|^3}), \tag{3.12}$$

whereas for *k=3,...,6* we will have

$$\omega^2 = \frac{1}{q'_k}(\mu_1 \sum_{j=1,j\neq k}^{2} \frac{q'_k - q_j}{|q'_k - q_j|^3} + \mu_2 \sum_{j=3,j\neq k}^{4} \frac{q'_k - q'_j}{|q'_k - q'_j|^3} + \mu_3 \sum_{j=5,j\neq k}^{6} \frac{q'_k - q'_j}{|q'_k - q'_j|^3}). \tag{3.13}$$

From the equalities (3.12) and (3.13), by condition (3.11), we obtain the following system of equations

$$\Psi_l(m, \mu_2, \mu_3, q, q') = 0, \quad l = 1,2,3. \tag{3.14}$$

Here, functions Ψ_l(m,μ_2,μ_3,q,q') have the form:

$$\begin{cases} \Psi_1 &= (\frac{1}{4q^3} - \frac{2}{(q^2+q'^2)^{\frac{3}{2}}})m + (-\frac{1}{4q'^3} + \frac{2}{(q^2+q'^2)^{\frac{3}{2}}})\mu_2 + \\ & (\frac{\sqrt{(q-q')^2}}{(q-q')^4} - \frac{q'\sqrt{(q-q')^2}}{q(q-q')^4} - \frac{1}{\sqrt{2}q'^3} + \frac{1}{(q+q')^3} + \frac{q'}{q(q+q')^3})\mu_3, \\ \Psi_2 &= (\frac{1}{4q^3} - \frac{\sqrt{(q-q')^2}}{(q-q')^4} - \frac{1}{(q+q')^3} + \frac{q\sqrt{(q-q')^2}}{q'(q-q')^4} - \frac{q}{q'(q+q')^3})m + \\ & (-\frac{1}{\sqrt{2}q'^3} + \frac{2}{(q^2+q'^2)^{\frac{3}{2}}})\mu_2 + \\ & (-\frac{1}{4q'^3} + \frac{\sqrt{(q-q')^2}}{(q-q')^4} + \frac{1}{(q+q')^3} - \frac{q'\sqrt{(q-q')^2}}{q(q-q')^4} + \frac{q'}{q(q+q')^3})\mu_3, \\ \Psi_3 &= (-\frac{\sqrt{(q-q')^2}}{(q-q')^4} - \frac{1}{(q+q')^3} + \frac{2}{(q^2+q'^2)^{\frac{3}{2}}} + \frac{q\sqrt{(q-q')^2}}{q'(q-q')^4} - \frac{q}{q'(q+q')^3})m + \\ & \frac{1-2\sqrt{2}}{4q'^3}\mu_2 + \frac{-1+2\sqrt{2}}{4q'^3}\mu_3. \end{cases} \tag{3.15}$$

The computations show that the determinant of the matrix of the system (3.14) is equal to zero. Using CAS Mathematica and Gauss' method we reduce the system (3.14) to the following one

$$\kappa_1 m + \kappa_2 \mu_2 + \kappa_3 \mu_3 = 0. \tag{3.16}$$

where the coefficients κ_1, κ_2, κ_3 are given by (3.3).

(<=) Let μ_2, μ_3 be known as (3.2) with the coefficients κ_s, *s*=1,2,3 for (3.4) and *m*> 0. These values satisfy the system (3.16) which in turn, is equivalent to (3.14). But recall that the equalities in (3.15) mean pair-wise equalities in (3.12) and (3.13), which through (3.11) implies that the equation (3.10) is satisfied. The last one is just the equation at central configuration.

Below we demonstrate some graphics for different values of parameters *m* and *q*. We can see that for given *m* and *q* the masses μ_2 and μ_3 are positive.

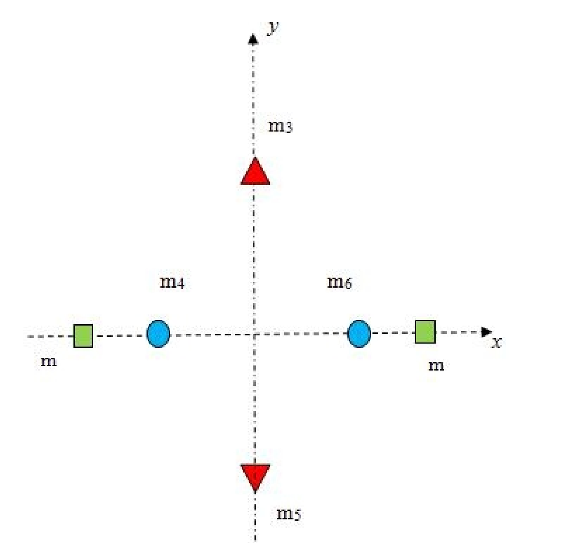

Figure 1. Configuration (0,4,2)

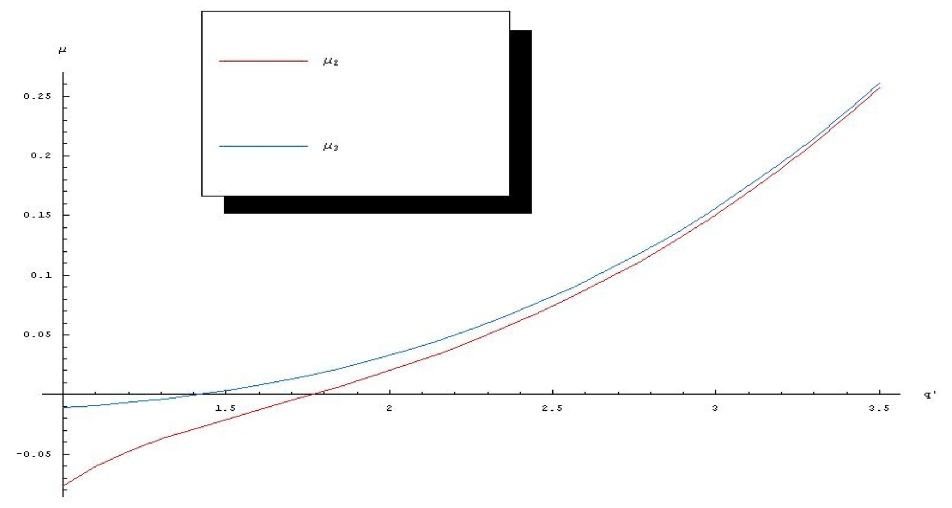

Figure 2. *m.=0,1 q=1*

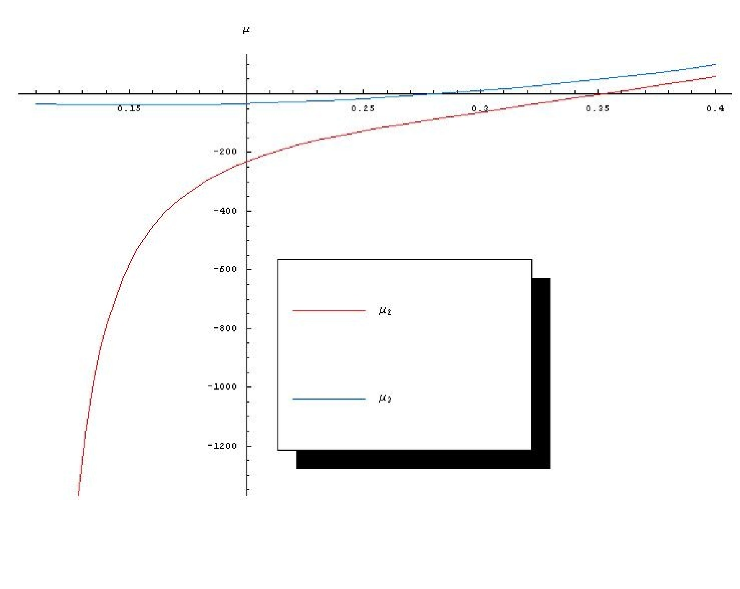

Figure 3. *m=0.5, q=1.5*

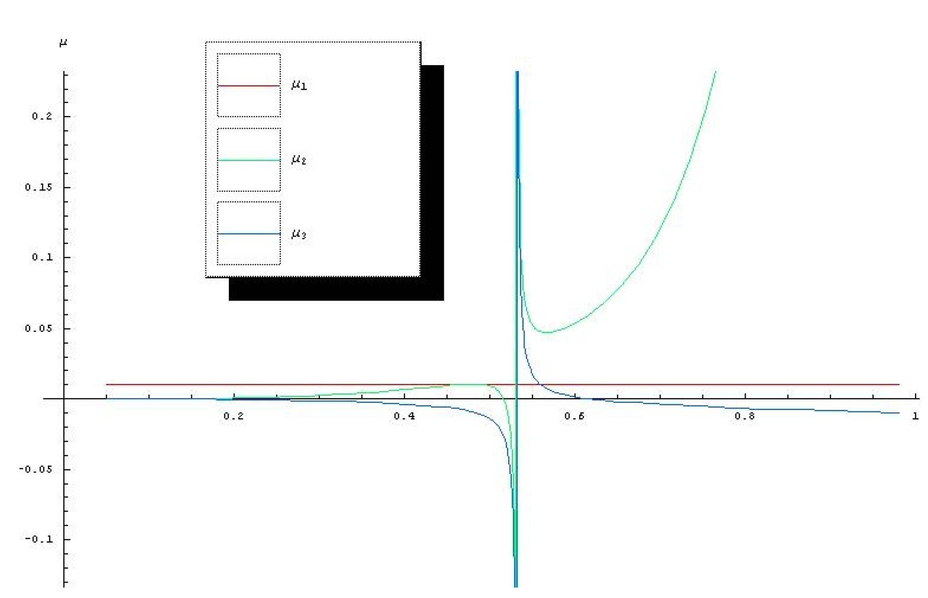

Figure 4 *m.=0.01, q=1*

References

[1] Arnold, V.I.,Kozlov, V.V., Neishtadt, A.I.: Mathematical Aspects of Classical and Celestial Mechanics. Springer-Verlag, Berlin (2006)

[2] Elmabsout, B.: Sur l'existence de certaines positions d'quilibre relatif dans le probleme des n corps. Celest. Mech., 41, (1988), 131-151

[3] Grebenicov, E.A.: Two new dynamical models in celestial mechanics. Rom. Astron. J., 10, No.1, (1998), 13-19

[4] Elmabsout, B.: Nouvelles congurations d'equilibre relatif pour le probleme des N corps. C. R. Acad. Sci., Serie II 312, (1991), 467-472.2

[5] Siluszyk, A.: On the linear stability of relative equilibria in the restricted eight-body problem with partial symmetry. Vesnik Brestcaga Universiteta, No.2, (2004), 20-26.

[6] Siluszyk, A.: On the relative equilibrium congurations in the planar ve-body problem. Opuscula Math., Ser.30, No.4, (2010), 495-506.

[7] Siluszyk, A.: New central congurations in the planar 6-body problem. Intern. Conf. on Nonlinear Operators, Dierent. Eq. and Appl., ICNODEA, July 5-8, 2011, Cluj- Napoca, Romania, Abstracts. Babes-Bolyai University Cluj-Napoca Press,p.84

[8]Grebenicov E.A., Kozak-Skoworodkin D., Jakubiak M. Computer Algebra Methods in the n-body problem, Moskwa, 2002 (in Russian).