A CLASSIFICATION OF EMBEDDINGS OF NON-SIMPLY CONNECTED 4-MANIFOLDS IN 7-SPACE

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Knotting Problem

The classical Knotting Problem runs as follows: *given an n-manifold N and a number m, describe the set of isotopy classes of embeddings $N \to \mathbb{R}^m$.* For recent surveys see [Sk08, MA2]; whenever possible we refer to these surveys not to original papers. We tacitly consider *smooth* embeddings and isotopies unless *piecewise linear (PL)* embeddings and isotopies are explicitly mentioned.

Figure 1: knotted torus in $\mathbb{R}^3$

The Knotting Problem is more accessible for $2m \geq 3n+4$ [Sk08, §2, §3, MA2]. It is much harder for

$$2m < 3n + 4 :$$

if N is a closed manifold that is not a disjoint union of homology spheres, then until recently no complete readily calculable descriptions of isotopy classes was known, in spite of the existence of interesting approaches of Browder-Wall and Goodwillie-Weiss [Wa70, GW99]. [1]

Recently there appeared two approaches allowing to classify embeddings for $2m < 3n+4$. One of them involves almost embeddings and β-invariant [Sk06, CRS07, CRS08], another uses the Kreck modified surgery

[1] The approach of [GW99] gives a modern abstract proof of certain earlier known results. We are grateful to M. Weiss for indicating that this approach also gives explicit results on higher homotopy groups of the space of embeddings $S^1 \to \mathbb{R}^n$.

[Kr99, Sk08', Sk10, CS11, CS]. Here we review applications of the latter approach to classification of embeddings of 4-manifolds into $\mathbb{R}^7$.

For a manifold N let $E^m(N)$ ($E^m_{PL}(N)$) be the set of smooth (PL) embeddings $N \to \mathbb{R}^m$ up to smooth (PL) isotopy.

Figure 2: connected sum group structure

Embedded connected sum

The 'connected sum' group structure on $E^m(S^n)$ was defined in [Ha66]. Haefliger proved that $E^m(S^n) = 0$ for $2m \geq 3n + 4$ [Sk08, §3]. However, $E^m(S^n) \neq 0$ for many m, n such that $2m < 3n + 4$, e.g. $E^7(S^4) \cong \mathbb{Z}_{12}$.

Figure 3: embedded connected sum

In this paragraph assume that N is a closed n-manifold and $m \geq n+3$. The group $E^m(S^n)$ acts on the set $E^m(N)$ by connected summation of embeddings $g : S^n \to \mathbb{R}^m$ and $f : N \to \mathbb{R}^m$ whose images are contained in disjoint cubes. [2] Various authors have studied the analogous connected sum action of the group of homotopy n-spheres on the set of smooth n-manifolds homeomorphic to given manifold; see for example [Le70]. For embeddings, the quotient of $E^m(N)$ modulo the above action of $E^m(S^n)$ is known in some cases. Thus in these cases the knotting problem is

[2] Since $m \geq n + 3$, the connected sum is well-defined, i.e. does not depend on the choice of an arc between gS^n and fN. If N is not connected, we assume that a component of N is chosen and we consider embedded connected summation with this chosen component.

reduced to *the determination of the orbits of this action.* This problem is just as difficult as the Knotting Problem: until recently *no results were known* on this action for $m \ge n+3$, $E^m(S^n) \ne 0$ and N not a disjoint union of spheres. For recent results see [Sk06, Sk08', Sk10, CS11, CS]; for a *rational* description see [CRS07, CRS08]; for $m = n+2$ see [Vi73].

Embeddings of 4-manifolds into $\mathbb{R}^m$

From now on N *is a closed connected orientable 4-manifold.* Since each smooth n-manifold has the PL structure compatible with the given smooth structure and each PL 4-manifold admits a unique smooth structure [Ma80, §1.2] we may consider N as simultaneously a smooth manifold and a PL manifold.

Until recently a complete, readily calculable classification of embeddings of N into $\mathbb{R}^m$ was only known for $m \ge 8$ (Wu and Haefliger-Hirsch) or for $N = S^4$ and $m = 7$ (Haefliger):

$$\#E^m(N) = 1 \quad \text{for} \quad m \ge 9, \quad E^8(N) = H_1(N; \mathbb{Z}_2), \quad E^7(S^4) \cong \mathbb{Z}_{12}.$$

It was also known that

- $E^m_{PL}(N) = E^m(N)$ for $m \ge 8$ and
- $E^m_{PL}(N) = H_2(N)$ when N is simply-connected [BH70].

Here the equality sign between sets denotes the existence of a bijection; the isomorphism is a group isomorphism for the 'connected sum' group structure. See references and more information in [MA1]. For a higher-dimensional analogue see [Ya84, Sk06, Sk10']. For embeddings of 4-manifolds with boundary into $\mathbb{R}^7$ see [To10].

In this paper we classify embeddings into $\mathbb{R}^7$ for general closed connected orientable torsion free 4-manifolds.

Main result for $\mathbb{C}P^2$ and $S^2 \times S^2$

Theorem 1. *There are exactly two isotopy classes of embeddings $\mathbb{C}P^2 \to \mathbb{R}^7$.*

The two isotopy classes are represented by the standard embedding and its composition with the symmetry w.r.t. $\mathbb{R}^6 \subset \mathbb{R}^7$. The standard embedding is given by

$$(x : y : z) \mapsto (x\overline{y}, y\overline{z}, z\overline{x}, 2|x|^2 + |y|^2), \quad \text{where} \quad |x|^2 + |y|^2 + |z|^2 = 1.$$

Addendum. *For each embeddings* $f : \mathbb{C}P^2 \to \mathbb{R}^7$ *and* $g : S^4 \to \mathbb{R}^7$ *embedding* $f\#g$ *is isotopic to* f.

Theorem 2. *For each integer* u *there are exactly* $GCD(u, 12)$ *isotopy classes of embeddings* $f : S^2 \times S^2 \to \mathbb{R}^7$ *with* $\varkappa(f) = (2u, 0)$, *and the same holds for those with* $\varkappa(f) = (0, 2u)$. *Other values of* $\mathbb{Z}^2$ *are not in the image of* $\varkappa$. *(We take the standard basis in* $H_2(S^2 \times S^2)$*;* $\varkappa$*-invariant is defined below.)*

Figure: construction of f_u

Construction of an embedding f_u *with* $\varkappa(f) = (2u, 0)$. Take the standard embeddings $2D^5 \times S^2 \subset \mathbb{R}^7$ (where 2 is multiplication by 2) and $\partial D^3 \subset \partial D^5$. Take u copies $(1 + \frac{1}{n})\partial D^5 \times x$ $(n = 1, \dots, u)$ of the oriented 4-sphere outside $D^5 \times S^2$ 'parallel' to $\partial D^5 \times x$. Join these spheres by tubes so that the homotopy class of the resulting embedding

$$S^4 \to S^7 - (D^5 \times S^2) \simeq S^7 - S^2 \simeq S^4 \quad \text{will be} \quad u \in \pi_4(S^4) \cong \mathbb{Z}.$$

Let f be the connected sum of this embedding with the standard embedding $\partial D^3 \times S^2 \subset \mathbb{R}^7$.

Addendum. *There are embeddings* $f_0, f_1 : S^2 \times S^2 \to \mathbb{R}^7$ *such that for each embedding* $g : S^4 \to \mathbb{R}^7$

- *embedding* $f\#g$ *is isotopic to* f.
- *embedding* $f\#g$ *is isotopic to* f *if and only if* g *is isotopic to the standard embedding.*

Main result for $S^1 \times S^3$

Theorem 3. *There is a commutative diagram of surjections*

$$\begin{tikzcd} \mathbb{Z}_{12} \times (\mathbb{Z} \times \mathbb{Z}) \arrow[r, "\# \times \tau"] \arrow[d, "0 \times (\mathrm{id} \times \rho_6)"] & E^7(S^1 \times S^3) \arrow[d, "\mathrm{forg}"] \\ \mathbb{Z} \times \mathbb{Z}_6 \arrow[r, "\tau_{PL}"] & E^7_{PL}(S^1 \times S^3) \end{tikzcd} \qquad \textit{such that}$$

$$a \# \tau(l,b) = a' \# \tau(l',b') \textit{ if and only if}$$

$$\left[\begin{array}{l} \textit{either} \quad l = l' \neq 0, \quad b \equiv b' \bmod 2l \quad \textit{and} \quad a = a', \\ \textit{or} \quad l = l' = 0, \quad b = b' \quad \textit{and} \quad a \equiv a' \bmod 2GCD(b,6) \end{array} \right. ,$$

$$\textit{and} \quad \tau_{PL}(l,b) = \tau_{PL}(l',b') \quad \textit{if and only if}$$

$$l = l' \quad \textit{and} \quad b \equiv b' \bmod 2GCD(l,3).$$

Here forg *is the forgetful map, τ is defined below and τ_{PL} is well-defined by $\tau_{PL}(l,b) := \operatorname{forg} \tau(l, \rho_6^{-1} b)$.*

Figure 4: construction of τ

Definition of τ. [Sk02] Identify $\pi_3(V_{4,2}) = \pi_3(S^3) \oplus \pi_3(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ by the standard isomorphisms. Take a map $\varphi : S^3 \to V_{4,2}$ representing $(l,b) \in \mathbb{Z} \oplus \mathbb{Z}$. By the exponential law we obtain a map $\varphi : S^3 \times S^1 \to \partial D^4$. Define the embedding $\tau(l,b)$ to be the composition

$$S^1 \times S^3 \overset{\varphi \times \mathrm{pr}_2}{\to} \partial D^4 \times S^3 \subset D^4 \times S^3 \subset \mathbb{R}^7.$$

An alternative construction of $\tau(1,0)$ and $\tau(0,1)$. These two embeddings are defined as compositions $S^1 \times S^3 \overset{\mathrm{pr}_2 \times t^i}{\to} S^3 \times S^3 \subset \mathbb{R}^7$, where $i = 1, 2$, pr_2 is the projection onto the second factor, $\subset$ is the standard inclusion and maps $t^i : S^1 \times S^3 \to S^3$ are defined below. We shall see that $t^i|_{S^1\times y}$ are embeddings for each $y \in S^3$, hence $\tau(1,0)$ and $\tau(0,1)$ are embeddings.

Define $t^1(s,y) := sy$, where S^3 is identified with the set of unit length quaternions and $S^1 \subset S^3$ with the set of unit length complex numbers.

Define $t^2(e^{i\theta},y) := H(y)\cos\theta + \sin\theta$, where $H : S^3 \to S^2$ is the Hopf map and S^2 is identified with the 2-sphere formed by unit length quaternions of the form $ai + bj + ck$.

In Theorem 3 the commutativity holds by definition and the surjectivity of forg is known [BH70]. The new and non-trivial part is the surjectivity and the description of orbits of $\# \times \tau$, as well as τ_{PL} being well-defined.

Theorem 3

• allows to disprove the Multiple Haefliger-Wu invariant conjecture, see the end of §1.

• disproves the Melikhov informal conjecture: *for each* $m \geq n+3$ *and a closed PL* n*-manifold* N *the set* $E^m_{PL}(N)$ *has a geometrically defined group structure.*

• shows that the parametric connected sum (see below) is well-defined and defines a group structure on $\mathrm{forg}^{-1}(\tau_{PL}(0,0))$;

• implies that the parametric connected sum (see below) is not well-defined for embeddings $S^1 \times S^3 \to \mathbb{R}^7$, both in the PL and the smooth cases.

Main results for the general case

In this paper we omit $\mathbb{Z}$-coefficients from the notation of (co)homology groups. We identify with $\mathbb{Z}$ the zero-dimensional homology group of a connected manifold.

Denote

• $H_k := H_k(N)$;

• by ρ_k the reduction modulo k.

• by $\#$ is the 'connected sum' action of $E^7(S^4)$ identified with $\mathbb{Z}_{12}$ (by the isomorphism η of [CS11]) on $E^7(N)$.

Theorem 4. *[Sk10], [CS11] Let N be a closed connected 4-manifold such that $H_1 = 0$. Then there is the Boéchat-Haefliger invariant*

$$\varkappa : E^7(N) \to H_2 \quad \text{whose image is}$$

$$\{u \in H_2 \mid \rho_2 u = PDw_2(N),\ u \cap u = \sigma(N)\}.$$

For each $u \in \operatorname{im}\varkappa$ there is a bijective invariant called the Kreck invariant,

$$\eta_u : \varkappa^{-1}u \to \mathbb{Z}_{\overline{u}}.$$

Here

$$\overline{u} := \begin{cases} GCD(u/2, 12) & \text{if } u \text{ is divisible by 2,} \\ GCD(u, 3) & \text{otherwise} \end{cases}.$$

Denote

- by $B(H_3)$ the space of bilinear forms $H_3 \times H_3 \to \mathbb{Z}$;
- by $\overline{l} : H_3 \to H_1$, for $l \in B(H_3)$, a homomorphism defined by $l(x, y) = x \cap \overline{l}(y)$;

For an element u of a free abelian group denote by $d(u)$ the divisibility of u, i.e. $d(0) = 0$ and $d(u)$ is the largest integer which divides u for $u \neq 0$.

For an integer d let $\overline{d}$ be $GCD(d/2, 12)$ or $GCD(d, 3)$ according to d divisible by 2 or not.

Theorem 5. *Let N be a closed connected orientable 4-manifold with torsion free H_1.*

(PL) There is a map

$$\varkappa_{PL} \times \lambda_{PL} : E^7_{PL}(N) \to H_2 \oplus B(H_3)$$

whose image consists of pairs (u, l) such that

$$\rho_2 u = PDw_2(N) \quad \text{and} \quad l(y, x) = l(x, y) + u \cap x \cap y \quad \text{for each} \quad x, y \in H_3.$$

For each $(u, l) \in \operatorname{im}(\varkappa_{PL} \times \lambda_{PL})$ there is a 1–1 map

$$\beta_{u,l,PL} : (\varkappa_{PL} \times \lambda_{PL})^{-1}(u, l) \to \frac{H_1}{2\overline{l}(H_3) + GCD(d(u), 6)H_1}.$$

(DIFF) There is a map

$$\varkappa\times\lambda : E^7(N)\to H_2\times B(H_3)$$

whose image consists of pairs (u,l) *such that* $u^2=\sigma(N)$,

$\rho_2 u = PDw_2(N)$ *and* $l(y,x)=l(x,y)+u\cap x\cap y$ *for each* $x,y\in H_3$.

For each $(u,l)\in\mathrm{im}(\varkappa\times\lambda)$ *there is a surjective map*

$$\beta_{u,l} : (\varkappa\times\lambda)^{-1}(u,l)\to\frac{H_1}{2\bar l(H_3)+d(u)H_1}$$

such that for each $b\in\mathrm{im}\,\beta_{u,l}$ *there is a 1–1 map*

$$\eta_{u,l,b} : \beta^{-1}_{u,l}(b)\to\mathbb{Z}_{\delta(u,l,b)},$$

where $\delta(u,l,b)$ *is an unknown divisor of* $\overline{d(u)}$.

(Addendum) Moreover, for each embeddings $g : S^4\to S^7$ *and* $f : N\to S^7$

$$\varkappa(f\#g)=\varkappa(f),\quad \lambda(f\#g)=\lambda(f),\quad \beta_f(f\#g)=\beta_f(f)\quad\textit{and}$$

$$\eta_f(f\#g)=\eta_f(f)+\rho_{\delta(f)}\eta(g),\quad\textit{where}\quad \beta_f:=\beta_{\varkappa(f),\lambda(f)},$$

$$\eta_f=\eta_{\varkappa(f),\lambda(f),\beta_f(f)}\quad\textit{and}\quad \delta(f)=\delta(\varkappa(f),\lambda(f),\beta_f(f)).$$

Maps $\lambda,\varkappa,\beta$ and η are defined below.

Note that

- Theorem 3 does not follow from Theorem 5.
- for each $u\in H_2$ the set $\lambda(\varkappa^{-1}(u))$ is in 1–1 correspondence with the set of *symmetric* bilinear forms $H_3\times H_3\to\mathbb{Z}$.

The multiple Haefliger-Wu invariant conjecture.

Borsuk-Ulam type ideas in the theory of embeddings lead to discovery of the Haefliger-Wu invariant. For a manifold N let

$$\widetilde N=\{(x_1,\dots,x_p)\in N^p\mid x_i\ne x_j\text{ for each }i,j\}.$$

The group S_p of permutations of p elements obviously acts on the space $\widetilde N$. For an embedding $f : N\to\mathbb{R}^m$ define the map

$$\widetilde f : \widetilde N\to\widetilde{\mathbb{R}^m}\quad\text{by}\quad \widetilde f(x_1,\dots,x_p)=(fx_1,\dots,fx_p).$$

Clearly, the map $\widetilde{f}$ is S_p-equivariant. Define the *multiple Haefliger-Wu invariant* $\alpha_p(f)$ to be the S_p-equivariant homotopy class of $\widetilde{f}$.

We disprove the following conjecture that was 'in the air' since 1960's (the authors learned it from A. N. Dranishnikov, E. V. Schepin, A. Szücs and O. Ya. Viro).

If the multiple Haefliger-Wu invariants of two PL embeddings of a connected PL n-manifold into $\mathbb{R}^m$ are equal and $m \geq n+3$, then the embeddings are PL isotopic.

Figure 5: the double Haefliger-Wu invariant

For more introduction see [Sk06].

Theorem 6. *For each p the multiple Haefliger-Wu invariant $\alpha_p : E^7_{PL}(S^1 \times S^3) \to [\widetilde{S^1 \times S^3}_p \to \widetilde{\mathbb{R}^7}_p]_{eq}$ is not injective, i.e. there exists PL embeddings $f, g : S^1 \times S^3 \to \mathbb{R}^7$ which are not PL isotopic but for which $\alpha_p(f) = \alpha_p(g)$.*

We can take $f = \tau(0,2)$ and $g = \tau(0,0)$ the standard embedding. By Theorem 3 f is not PL isotopic to g. The new β-invariant allowing to distinguish embeddings these was constructed in frame of the Kreck surgery approach to classification of embeddings. However, the main ideas of proof could be clearly presented to non-specialists.

The *non-manifold* example of incompleteness of the Haefliger-Wu obstruction to embeddability (Segal, Spiez, Freedman, Krushkal, Teichner and the author, 1990s) were cleverly used to obtain some algorithmic results on the *existence* of embeddings (Matoušek, Tancer, Wagner, 2008). It would be interesting to know whether the new *manifold* counterexample could be used to obtain algorithmic results on *distinguishing* embeddings.

Main definitions and notation

We tacitly consider only *compact* manifolds.

Throughout this paper N is a closed connected orientable 4-manifold with torsion free H_1 and $f : N \to S^7$ is an embedding. Fix an orientation on N and an orientation on S^7. Denote by

• $N_0 := \mathrm{Cl}(N - B^4)$, where B^4 is a closed 4-ball in N.

• $C = C_f$ the closure of the complement in $S^7 \supset \mathbb{R}^7$ to a tubular neighborhood of $f(N)$; the orientation on C is inherited from the orientation of S^7.

• $\nu = \nu_f$ the normal vector bundle of f.

In this and other notation we sometimes omit subscript f.

Figure 6: main objects

Identify ∂C with the total space of the sphere bundle of ν. In this paper a bundle isomorphism is always the restriction to the sphere bundle of a linear bundle isomorphism identical on the base.

We denote Poincaré duality by PD. We denote the maps induced in homology by the same letters as inducing maps (no confusion would arise). The homology intersection products in N are denoted by $\cap$, and intersection products in other spaces have subscripts indicating the space. The well-known definitions of such products are recalled in [Sk10, Remark 2.3].

Let $[N] \in H_4$ be the fundamental class of N. For a manifold P denote $H_s(P, \partial) := H_s(P, \partial P)$. For a map $\xi : P \to Q$ between a p-manifold and a q-manifold denote the 'preimage' homomorphism by

$$\xi^! := PD \circ \xi^* \circ PD : H_s(Q, \partial) \to H_{p-q+s}(P, \partial).$$

For a q-manifold Q, a p-submanifold $P \subset Q$ (possibly, with boundary and possibly $p = q$) and either $y \in H_s(Q)$ or $y \in H_s(Q, \partial)$ denote

$$r_{Q,P}(y) = r_P(y) = y \cap P := PD[(PDy)|_P] \in H_s(P, \partial).$$

If y is represented by a closed oriented submanifold $Y \subset Q$ transverse to P, then $y \cap P$ is represented by $Y \cap P$. We use the same notation for any coefficient group.

We denote by $i_{P,X}, j_{P,X}, \partial_{P,X}$ or shortly by i_P, j_P, ∂_P the homomorphisms from the exact sequence of pair (P, X). If $P = C_{f_k}$ or $P = C_f$, then we shorten the subscript C_{f_k} or C_f to just k or just f, respectively.

By e we denote any excision isomorphism.

Definitions of $\varkappa$- and λ-invariants

Let $\zeta : N_0 \to \nu^{-1}N_0$ be a section of the normal bundle $\nu^{-1}N_0 \to N_0$. (This exists because the Euler class of ν is zero.) Consider the following diagram.

$$H_4(N_0, \partial) \xrightarrow{\zeta} H_4(\nu^{-1}N_0, \partial) \xleftarrow[e]{} H_4(\partial C, \nu^{-1}B^4) \xleftarrow[j_{\partial C}]{} H_4(\partial C) \xrightarrow{i_C} H_4(C)\,.$$

Here $j_{\partial C}$ is an isomorphism. Section ζ is called *weakly unlinked* if $i_C j_{\partial C}^{-1} e^{-1}\zeta = 0$.

E.g. any section $N_0 \to \nu^{-1}N_0$ is weakly unlinked for $N = S^1 \times S^3$ because $H_4(S^7 - S^1 \times S^3) \cong H_2(S^1 \times S^3) = 0$.

A weakly unlinked section exists and is unique up to equivalence over 2-skeleton of N (by [BH70, Proposition 1.3], cf. [Sk08', the Unlinked Section Lemma (a)], because by [Sk10, Remark 2.4 and footnote 14] our definition of a weakly unlinked section is equivalent to the original definition [BH70]).

Definition of the Boéchat-Haefliger invariant $\varkappa : E^7(N) \to H_2$. Represent a class $x \in H_2$ by closed oriented 2-submanifolds (or integer 2-chain) $X \subset N_0$. Take a weakly unlinked section $\xi : N_0 \to \partial C$. Since H_2 has no torsion, $\varkappa(f)$ is uniquely defined by

$$\varkappa(f) \cap x := \mathrm{lk}_{S^7}(fN, \xi X).$$

This is well-defined, i.e., is independent of ξ, by [CS]. This definition is equivalent to those of [BH70], [Sk10], [CS11] by [CS].

Definition of Seifert form $\lambda : E^7(N) \to B(H_3)$. Represent classes $x, y \in H_3$ by closed oriented 3-submanifolds (or integer 3-chains) $X, Y \subset N_0$. Take a weakly unlinked section $\xi : N_0 \to \partial C$. Define

$$\lambda(f)(x, y) = \mathrm{lk}_{\,S^7}(fX, \xi Y) \in \mathbb{Z}.$$

This is well-defined, i.e., is independent of ξ, by [CS] or else by [To10].

Clearly, $\lambda(f) : H_3 \times H_3 \to \mathbb{Z}$ is indeed a bilinear form.

Remarks. (a) If in definitions of $\varkappa$ and λ we take an arbitrary (i.e. not weakly unlinked) section ξ, we obtain different values.

(b) Although a weakly unlinked section is defined over N_0, its construction essentially uses f not only $f|_{N_0}$. For embeddings $N_0 \to S^7$ the Boéchat-Haefliger invariant does not exist and the definition of Seifert form is more complicated, cf. [To10].

(c) Weakly unlinked sections may differ on 3-skeleton, and it seems that change of ξ on 3-skeleton may change $\mathrm{lk}_{S^7}(fX, \xi Y)$. The change is however trivial. Formal explanation for this is given above. Informally, the change is trivial because it is 'factored' through $H_3(S^2_f) = 0$.

Figure 7: definition of $\varkappa$- and λ-invariants

Lemma 7. *(simplification)* $\lambda(f)(x, y) = \mathrm{lk}_{S^7}(fX, fY)$ *if classes x and y are represented by disjoint closed oriented 3-submanifolds (or integer 3-chains) X and Y.*

($\varkappa$-symmetry) $\lambda(f)(y, x) = \lambda(f)(x, y) - \varkappa(f) \cap x \cap y$.

By the simplification $\lambda(f) = \mathrm{lk}(f|_{1\times S^3}, f|_{\{-1\}\times S^3})$ for an embedding $f : S^1 \times S^3 \to \mathbb{R}^7$. So the matrix of $\lambda\tau(l, b)$ in the canonical basis is (l).

By definition it is clear that $\lambda(f) = \varkappa(f) = 0$ when $f(N) \subset \mathbb{R}^6$.

Conjecture 8. If $f : N \to \mathbb{R}^7$ is an embedding such that $f(N) \subset \mathbb{R}^6$ and embeddings $g_1, g_2 : S^4 \to \mathbb{R}^7$ are not isotopic, then $f\#g_1$ and $f\#g_2$ are not isotopic.

This is only proved in [Sk10, The Effectiveness Theorem 1.2] when ΣN retracts to $\Sigma\,\mathrm{Cl}(N - B^4)$, e.g. $N = S^1 \times S^3$ or N spin simply-connected.

Definition of β-invariant

In this subsection $f_0, f_1 \in (\lambda \times \varkappa)^{-1}(l, u)$ and $\varphi : \partial C_0 \to \partial C_1$ is an orientation-preserving bundle isomorphism. In the sequel $k = 0, 1$; we change subscripts 'f_k' to 'k'. Denote

$$M = M_\varphi := C_0 \cup_\varphi (-C_1).$$

Recall that a *(stable tangent) string structure* on a manifold is a stable tangent framing on 3-skeleton (of some triangulation) extendable to 4-skeleton, up to isotopy on 3-skeleton. Recall that a *stable tangent spin structure* on a manifold is a stable tangent framing on 1-skeleton (of some triangulation) extendable to 2-skeleton, up to isotopy on 1-skeleton. A bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$ is called *string* if it carries the string structure on ∂C_0 coming from S^7 to the string structure on ∂C_1 coming from S^7.

Lemma 9. *A string bundle isomorphism exists and is unique (up to equivalence) over N_0.*

Note that for this lemma we do not need that $\varkappa(f_0) = \varkappa(f_1)$ and $\lambda(f_0) = \lambda(f_1)$.

Figure 8: definition of β-invariant

Take a small oriented disk $D^3_f \subset \mathbb{R}^7$ whose intersection with fN consists of exactly one point of sign $+1$ and such that $\partial D^3_f \subset \partial C_f$. *The* *meridian* $S^2_f \in H_2(C_f)$ of f is the homology class of ∂D^3_f. By Alexander duality S^2_f generates $H_2(C_f)$.

A class $Y \in H_5(M_\varphi;\mathbb{Z}_d)$ such that $Y \cap i_{M_\varphi} S^2_{f_0} = 1$ is called a *joint ($\mathbb{Z}_d$-)homology Seifert surface (for φ)*.

Lemma 10. *Let $\varphi : \partial C_0 \to \partial C_1$ be a string bundle isomorphism.*

(a) For each $d \in \mathbb{Z}$ there is a joint homology Seifert surface $Y \in H_5(M_\varphi;\mathbb{Z}_d)$

(b) For each joint homology Seifert surface $Y \in H_5(M_\varphi;\mathbb{Z}_d)$ there is a unique $b \in H_1(N;\mathbb{Z}_d)$ such that $i_{M_\varphi}\nu^!_{f_0} b = Y^2 \in H_3(M;\mathbb{Z}_d)$.

(c) (Well-definition for β)

$$\beta(f_0, f_1) := [(i_{M_\varphi}\nu^!_{f_0})^{-1}Y^2] \in \frac{H_1}{2\bar{l}(H_3) + d(u)H_1}$$

is independent of the choice of a string bundle isomorphism φ and a joint $\mathbb{Z}$-homology Seifert surface Y for φ.

(d) If $\beta(f_0, f_1) = 0$, then there is a joint homology Seifert surface $Y \in H_5(M_\varphi)$ such that $Y^2 \in d(u)H_3(M_\varphi)$.

A class $Y \in H_5(M_\varphi)$ as in Lemma 10.d is called *a faithful class (for φ).*

Definition of η-invariant

Definition of $\eta(f_0, f_1, Y)$. Assume that $\beta(f_0, f_1) = 0$. Take a string bundle isomorphism $\varphi : \partial C_0 \to \partial C_1$. By Lemma 10.d there is a faithful class $Y \in H_5(M_\varphi)$. By the proof of [CS11], Null-bordism Lemma 2.6, there is a compact connected spin 8-manifold W and $z \in H_6(W, \partial)$ such that $\partial W \underset{spin}{=} M_\varphi$ and $\partial z = Y$. Denote $d := d(u)$. Consider the following exact sequence of pair (W, M_φ):

$$H_4(M_\varphi;\mathbb{Z}_d) \stackrel{i_W}{\to} H_4(W;\mathbb{Z}_d) \stackrel{j_W}{\to} H_4(W, M_\varphi;\mathbb{Z}_d) \stackrel{\partial_W}{\to} H_3(M_\varphi;\mathbb{Z}_d).$$

Since

$$\partial_W \rho_d z^2 = \rho_d Y^2 = 0, \quad \text{there is} \quad \overline{z^2} \in H_4(W;\mathbb{Z}_d) \quad \text{such that} \quad j_W \overline{z^2} = \rho_d z^2.$$

Define

$$\eta(f_0, f_1, Y) := \rho_{\overline{d}} \frac{\overline{z^2} \cap (z^2 - p_W)}{2}.$$

Proof that $\overline{z^2} \cap (z^2 - p_W) \in \mathbb{Z}_d$ is divisible by 2 for d even. The spin structure on W together with $z \in H_6(W, \partial) \cong H^2(W) \cong [W, \mathbb{C}P^\infty]$ define a map $W \to B\,\mathrm{Spin} \times \mathbb{C}P^\infty$. By surgery of this map relative to the boundary we may assume that it is 3-connected. The number $\overline{z^2} \cap (z^2 - p_W)$ does not change throughout this surgery because it is '$B\,\mathrm{Spin} \times \mathbb{C}P^\infty$-characteristic number'. Since the map is 3-connected, we have $\mathrm{Tors}\, H_4(W) \cong \mathrm{Tors}\, H_3(W) = 0$. Hence there is $\widehat{z^2} \in H_4(W)$ such that $\rho_d \widehat{z^2} = \overline{z^2}$. Then

$$\overline{z^2} \cap (z^2 - p_W) = \rho_d(\widehat{z^2} \cap z^2 - \widehat{z^2} \cap p_W) = \rho_d(\widehat{z^2} \cap \widehat{z^2} - \widehat{z^2} \cap p_W).$$

The latter expression is divisible by 2 by [CS11, Lemma 2.11]. $\square$

Figure 9: definition of η-invariant

Proof that $\eta(f_0, f_1, Y)$ is well-defined, i.e., is independent of the choice of The proof for $\overline{z^2}, W, z$ is analogous to [CS11, 2.3]. For $\overline{z^2}$ instead of [CS11, Lemma 2.7] we use that $\partial_W p_W = p_{M_\varphi} = 0$. For W, z instead of the uniqueness of $\partial_W z$ of [CS11, Lemma 2.6] we use that $\partial_W z = Y$ is fixed.

The proof for φ is analogous to [CS11, Framing Theorem 2.9.φ]. Instead of $H_3 = 0$ we use that φ is string not only spin. $\square$

Lemma 11 (Canonical embedding)**.** *For each $(u, l) \in \mathrm{im}(\varkappa \times \lambda)$ there is an embedding $f_{u,l} \in (\varkappa \times \lambda)^{-1}(u, l)$ such that $\eta(f_{u,l}, f_{u,l}, Y) = 0 \in \mathbb{Z}_{\overline{d(u)}}$ whenever Y is a faithful class for* $\mathrm{id}\, \partial C_{f_{u,l}}$.

Lemma 12 (Well-definition for η)**.** *Assume that $f_0, f_1 \in (\varkappa \times \lambda)^{-1}(u, l)$ and $\beta(f_0, f_1) = 0$.*

(a) The residue

$$\eta(f_0, f_1) := \rho_{\delta(u,l,\beta(f_0, f_{u,l}))} \eta(f_0, f_1, Y) \in \mathbb{Z}_{\delta(u,l,\beta(f_0, f_{u,l}))}$$

is independent of a faithful class Y.

(b) If $\eta(f_0, f_1) = 0$, then there is a faithful class Y such that $\eta(f_0, f_1, Y) = 0 \in \mathbb{Z}_{\overline{d(u)}}$.

Properties of the invariants

Lemma 13 (Transitivity). *For each three embeddings $f_0, f_1, f_2 : N \to S^7$ having the same values of $\varkappa$- and λ-invariants*

$$\beta(f_2, f_0) = \beta(f_2, f_1) + \beta(f_1, f_0)$$

$$\textit{and, if} \quad \beta(f_0, f_1) = \beta(f_1, f_2) = 0, \quad \eta(f_2, f_0) = \eta(f_2, f_1) + \eta(f_1, f_0).$$

Lemma 14 (Additivity). *For each embeddings $g : S^4 \to S^7$, $f : N \to S^7$ and the standard embedding $g_0 : S^4 \to S^7$*

$$\varkappa(f\#g) = \varkappa(f), \quad \lambda(f\#g) = \lambda(f), \quad \beta(f\#g, f) = 0 \quad \textit{and}$$

$$\eta(f\#g, f) = \rho_{\delta(f)}\eta(g, g_0).$$

Lemma 15 (Isotopy classification). *If $\lambda(f_0) = \lambda(f_1)$, $\varkappa(f_0) = \varkappa(f_1)$, $\beta(f_0, f_1) = 0$ and $\eta(f_0, f_1) = 0$, then f_0 is isotopic to f_1.*

Figure 10: parametric connected sum

In our proof we use extensively parametric connected sum [Sk07], [MA], in spite of its not being well-defined for embeddings of 4-manifolds into $\mathbb{R}^7$.

Lemma 16 (Parametric additivity). *For each $p \in H_1$ and each embedding $h := f +_p \tau(l, b)$*

$$\varkappa(h) = \varkappa(f), \quad \lambda(h)(y, z) = \lambda(f)(y, z) + l(p \cap y)(p \cap z),$$

$$\textit{and,} \quad \textit{for } l = 0, \quad \beta(h, f) = [bp].$$

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