Identities in varieties generated by algebras of subalgebras

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WYDZIAŁ MATEMATYKI I NAUK INFORMACYJNYCH

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BASIC CONCEPTS

Power algebras of SUBSETS

Let (A,Ω) be an algebra. Consider $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A. For any n-ary operation $\omega\colon A^n \to A$ we define **the complex** (**or power**) **operation**

$$\omega\colon (\mathcal{P}_{>0}A)^n \to \mathcal{P}_{>0}A$$

in the following way:

$$\omega(A_1,\dots,A_n) := \{\omega(a_1,\dots,a_n) \mid a_i \in A_i\}.$$

We obtain $(\mathcal{P}_{>0}A,\Omega)$: **the power** (**complex** or **global**) **algebra** of an algebra (A,Ω).

EXAMPLES

1. **Groups:** multiplication of cosets of a subgroup of a group introduced by Frobenius;
2. **Lattices:** the set of ideals of a distributive lattice $(L,\vee,\wedge)$ again forms a lattice, where meets and joins are precisely the power operations of $\vee$ and $\wedge$;
3. **Formal languages:** the product of two languages is the power operation of concatenation of words.

Power algebras of SUBALGEBRAS

Let (A,Ω) be an algebra. Let AS be the set of all (non-empty) subalgebras of (A,Ω). In general, the family AS has not to be closed under complex operations. However if it does, (AS,Ω) is a subalgebra of the algebra $(\mathcal{P}_{>0}A,\Omega)$ and is called **the algebra of subalgebras** of (A,Ω).

Properties

Not all properties of an algebra (A,Ω) remain invariant under power construction!

In particular, not all identities true in (A,Ω) will be satisfied in $(\mathcal{P}_{>0}A,\Omega)$ or in (AS,Ω), e.g. **the power algebra of a group is not again a group** [1].

POWER ALGEBRAS OF SUBSETS

Let $\mathcal{V}$ be an arbitrary variety of algebras, let $\mathcal{V}\Sigma$ denote the variety generated by power algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{V}\Sigma := \mathrm{HSP}(\{(\mathcal{P}_{>0}A,\Omega) \mid (A,\Omega)\in\mathcal{V}\}).$$

G. Grätzer and H. Lakser [1] determined **the identities satisfied by the variety** $\mathcal{V}\Sigma$ in relation to identities true in $\mathcal{V}$.

We call a term t of the language of a variety $\mathcal{V}$ **linear**, if every variable symbol occurs in t at most once. An identity $t\approx u$ is called **linear**, if both terms t and u are linear.

EXAMPLES Let $\mathcal{V}$ be the variety of all algebras with one binary operation $\bullet$. The identity:
$x\bullet(y\bullet z)=(x\bullet y)\bullet z$ is **linear**, while
$x\bullet(y\bullet z)=(x\bullet z)\bullet(y\bullet z)$ is **not**.

THEOREM 1 *[1] Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}\Sigma$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.*

OUR AIM: Extend the result above for algebras of subalgebras.

POWER ALGEBRAS OF SUBALGEBRAS

Assume that for every algebra in $\mathcal{V}$, its algebra of subalgebras is defined. Denote by $\mathcal{VS}$ the variety generated by algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathcal{VS} := \mathrm{HSP}(\{(AS,\Omega) \mid (A,\Omega)\in\mathcal{V}\}).$$

Reasonable class to consider is a variety of **MODES** - algebras which are both **idempotent** and **entropic**.

IDEMPOTENT = each singleton is a subalgebra, or for each operation ω in Ω

$$\omega(x,\dots,x)\approx x.$$

ENTROPIC = any two of operations commute, or any operation is a homomorphism, or for every m-ary $\omega\in\Omega$ and n-ary $\phi\in\Omega$

$$\omega(\phi(x_{11},\dots,x_{n1}),\dots,\phi(x_{1m},\dots,x_{nm}))\approx\phi(\omega(x_{11},\dots,x_{1m}),\dots,\omega(x_{n1},\dots,x_{nm})).$$

Modes were introduced and investigated in detail by A. Romanowska and J.D.H. Smith [5].

EXAMPLES: affine spaces, convex sets, semilattices, normal bands.

WHY MODES? For a mode its algebra of subalgebras is always defined and moreover is again a mode [5].

HYPOTHESIS

Characterization as in Theorem 1 for varieties $\mathcal{VS}$ is still **NOT KNOWN**. Attempts to solve started in 1990s.

More about MODES

Let $\mathcal{M}$ be a variety of modes.

1. Entropic identities are linear so the variety $\mathcal{M}\Sigma$ is. But it is very rarely idempotent.
2. $\mathcal{M}\subseteq\mathcal{MS}\subseteq\mathcal{IM}\subseteq\mathcal{M}\Sigma$, where $\mathcal{IV}$ is the **idempotent subvariety of** $\mathcal{V}\Sigma$.

CONJECTURE 2 *An idempotent variety $\mathcal{V}$, in which every algebra has the algebra of subalgebras, coincides with $\mathcal{VS}$ if and only if $\mathcal{V}$ has a basis consisting of idempotent and linear identities.*

For non-idempotent varieties is **FALSE**!

$\Longleftarrow$ If a variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{VS}\subseteq\mathcal{V}$. Hence, if an idempotent variety $\mathcal{V}$ is defined by a set of linear identities, then $\mathcal{VS}=\mathcal{V}$.

??? $\Longrightarrow$ Though $\mathcal{VS}$ satisfies the linear identities true in $\mathcal{V}$, it is usually very difficult to determine which non-linear identities true in $\mathcal{V}$ are also satisfied in $\mathcal{VS}$.

MAIN RESULT

THEOREM 3 *[4] Let $\mathcal{M}$ be a variety of modes and let the variety $\mathcal{M}\Sigma$ be locally finite. The variety $\mathcal{MS}$ satisfies precisely the consequences of the idempotent and linear identities true in $\mathcal{M}$.*

PROOF based on the observation that any quotient of a mode (M,Ω) from locally finite variety $\mathcal{M}\Sigma$ that lies in $\mathcal{IM}$ actually lies in the subvariety $\mathcal{MS}$.

- Take a mode (M,Ω).
- Make the power algebra (of finite subsets): $(\mathcal{P}^{<\omega}_{>0}M,\Omega)$.
- Extend the language, i.e. add set-theoretical union: $(\mathcal{P}^{<\omega}_{>0}M,\Omega,\cup)$.
- Take the least congruence which give an idempotent factor. Here it is the congruence α which glues together subsets which generate the same subalgebras of M.
- Make a factor algebra $(\mathcal{P}^{<\omega}_{>0}M/\alpha,\Omega,\cup)$.
- Restrict the language, i.e. forget about $\cup$.
- Consider the variety
$$\mathrm{HSP}(\{(\mathcal{P}^{<\omega}_{>0}M/\alpha,\Omega) \mid (M,\Omega)\in\mathcal{M}\}).$$
It coincides with $\mathcal{MS}$. It also satisfies only linear and idempotent identities.

Algorithm 1: *SKETCH OF PROOF*

Earlier attempts to prove CONJECTURE provided only some partial solutions and applied the techniques proposed by G. Grätzer and H. Lakser [1]. Our solution used a completely different approach.

CONCLUSIONS

Local finiteness is CRUCIAL in the proof!

QUESTION Is the result true in more general case?

HOPE We know examples where the conclusion of the theorem holds, while the variety of power algebras is not locally finite.

References

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